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NESS Theory of Random Steady States*

T. Keyes[1,2]

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A theory is presented for the properties of random steady states based upon a generalization of existing theories of nonrandom steady states. A sample calculation is presented for the "energy cascade" in a weakly stirred system. The theory introduces multivariable time correlation functions, and a new method for evaluating such objects is also given.

KEY WORDS: Steady state; energy cascade.

1. INTRODUCTION

Recently[1-7] there has been a surge of interest in the properties of steady state (ss) systems. Several approaches are available in this area. We prefer, and will use, that developed[1-5] by the MIT group, the "NESS" (nonequilibrium steady state) formalism. In a ss, the distribution function may be written as the local equilibrium distribution plus a correction; the correction yields new, often surprising effects. In existing theories, the correction is given as a function of the gradients of the usual thermodynamic potentials, ϕ, the chemical potential, velocity field, and temperature, which characterize the ss and are assumed known. Random ss, however, defy such characterization. For example, the "cascade effect"[8] may be illustrated by[8] a system randomly stirred at low wave vector; one attempts to calculate the kinetic energy at intermediate and high wave vector, which arrives via "cascade" from the stirring. The calculation is very difficult and has been an outstanding problem in statistical physics for years. A randomly stirred system may define a ss, via time or ensemble averages, but it

* Research supported by NSF grant No. CHE78-09704.

[1] Sterling Chemistry Laboratory, 225 Prospect Street, New Haven, Connecticut 06511.
[2] Dreyfus Teacher—Scholar.

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is spatially uniform on average; no gradients in thermodynamic potentials (conjugate to the averages) exist.

It is desirable that the new, systematic methods[1-5] for treatment of ss be applied to random systems. Such application is the goal of this paper. In Section 2, we give two approaches to the theory of random ss, both based on the NESS formalism. Under the most simple approximations, both approaches are equivalent. Our general conclusions are stated in Section 3. As an illustration of the theory, we then show, in the remainder of the paper, how it can be used to study a randomly stirred system.

2. THEORY OF RANDOM NESS[3]

The NESS formalism requires, as do most dynamical theories, identification of the "slowly varying variables" in the system; these will be denoted A. The usual choice for A is that of the conserved densities, number, momentum, and energy, but, as we shall see, extended definitions of A may be useful. For now we retain the usual choice. The ss average of any quantity x, $\langle x\rangle_{ss}$, may then[1-5] be written as a sum of a local equilibrium part and a dissipative part,

$$\langle x\rangle_{ss} = \langle x\rangle_{LE} + \langle x\rangle_{D} \tag{1}$$

where $\langle\ \rangle_{LE}$ denotes an average with the local equilibrium distribution function,

$$f_{LE} = f_{eq}\exp\left[\ \beta\int d\mathbf{r}\, A(\mathbf{r})\phi(\mathbf{r})\ \right] \tag{2}$$

f_{eq} is the equilibrium distribution, ϕ is a column vector of thermodynamic forces conjugate to the variables A, and $\beta = (k_B T_{eq})^{-1}$, k_B being Boltzmann's constant and T_{eq} the equilibrium temperature $[T(\phi = 0)]$. In the usual case, the ϕ are the chemical potential, velocity field, and temperature. The spatial integral in Eq. (2), and in the following, is over the entire system.

The dissipative term, $\langle x\rangle_D$, is given by the relation

$$\langle x\rangle_{D} = -\int_0^\infty dt\int d\mathbf{r}\left\langle x(t)\dot{A}^{D}(\mathbf{r})\right\rangle_{LE}\phi(\mathbf{r}) \tag{3}$$

where $\dot{\mathbf{A}}^D$ is the dissipative time derivative,

$$\dot{A}^{D}(\mathbf{r}) = \dot{A}(\mathbf{r}) - MA(\mathbf{r}) \tag{4}$$

and M is the hydrodynamic matrix, which describes the assumed slow

[3] For simplicity of notation, we supress vector symbols when referring to vectors in dynamical variable space.

variation of the A. When the A are the conserved densities, M is a differential operator which generates the ordinary linearized hydrodynamic equations. In this case, we may write

$$\dot{A}^D(\mathbf{r}) = \nabla \cdot \mathbf{I}^D(\mathbf{r}) \tag{5}$$

which defines the dissipative flux, I^D. Equation (3) may then be transformed,

$$\langle x \rangle_D = \int_0^\infty dt \int d\mathbf{r} \langle x(t) \mathbf{I}^D(r) \rangle \cdot \nabla \phi(\mathbf{r}) \tag{6}$$

In a random ss, spatially uniform on average, $\nabla\phi$ vanishes. A process such as viscous heating in a stirred system may change the value of ϕ while maintaining spatial uniformity, but the resulting system is[1–5] just a different equilibrum system, so no new features are present and the phenomenon is uninteresting to us. Eqs. (1)–(3) are trivial under these circumstances, with $\langle x \rangle_{LE}$ always an equilibrium average and $\langle x \rangle_D$ zero; this is the difficulty discussed in the Introduction.

How, then, can we obtain $\langle x \rangle_{ss}$? First consider an intuitive "adiabatic" approach. If the random external force driving the system changes very slowly with time, we may consider that the random system passes in time from one nonuniform NESS to another, with a corresponding time-varying set of ϕ's. The ss average is then a composite of the average in Eq. (1) and an average, denoted by an overbar, over the fluctuating ϕ's with $\overline{\phi} = 0$ for a random system. With this viewpoint, the definition of ϕ is extended, and is related to an instantaneous average rather than a full average, which is surely reasonable for slow time variations. Thus,

$$\langle x \rangle_{ss} = \overline{\langle x \rangle}_{LE} + \overline{\langle x \rangle}_D \tag{7}$$

Equation (7) greatly simplifies if the ss remains near equilibrium, i.e., if the ϕ are small. Thus, we expand $\langle x \rangle_{LE}$ and $\langle x \rangle_D$ in ϕ, note that $\overline{\phi} = 0$, and keep the first nonzero term, that quadratic in ϕ, with the result

$$\langle x \rangle_{ss} = \int d\mathbf{r}\, d\mathbf{r}' \Big\{ 1/2 \langle xA(\mathbf{r})A(\mathbf{r}') \rangle : \overline{\phi(\mathbf{r}')\phi(\mathbf{r})} + \int_0^\infty dt \, \langle x(t) A(\mathbf{r}') \mathbf{I}^D(\mathbf{r}) \rangle \cdot \nabla_{\mathbf{r}} \overline{\phi(\mathbf{r})\phi(\mathbf{r}')} \Big\} \tag{8}$$

where an unsubscripted average denotes equilibrium. Equation (8) represents an intuitive attempt to use the NESS formalism for slowly fluctuating random ss weakly displaced from equilibrium. Just as the properties of nonrandom ss near equilibrium are determined by $\nabla\phi$ (assumed known), here we must assume that the correlation function of the random thermody-

namic force, $\overline{\phi(\mathbf{r})\phi(\mathbf{r}')}$, is known. Such correlation functions are familiar quantities, including, e.g., the velocity correlation function of turbulence theory. As we shall see, the most interesting features of random ss come from the dissipative average (just like the nonrandom case), so, more precisely, it is the gradient of the correlation function which replaces the gradient of ϕ in the central role; the random external force generates nonequilibrium spatial correlations which "drive" the system.

The problem can be approached more formally. In his thesis,[5] Machta noted that, in the usual definition of local equilibrium, higher moments of the A's cannot be adjusted independently from $\langle A\rangle_{\mathrm{LE}}$; all is determined by ϕ. He proposed a "generalized local equilibrium" (GLE) where the usual ϕ, now denoted $\phi^{(1)}$, fixes $\langle A\rangle_{\mathrm{LE}}$, but newly defined $\phi^{(n)}$ determine the higher moments, and

$$f_{\mathrm{GLE}} = f_{\mathrm{eq}} \exp\left[\beta \sum_{n=1}^{m} d\mathbf{r}_1 \cdots d\mathbf{r}_n \phi^{(n)}(\mathbf{r}_1 \cdots \mathbf{r}_n) A^{(n)}(\mathbf{r}_1 \cdots \mathbf{r}_n)\right] \tag{9}$$

where m is to be chosen according to the demands of a particular problem. Close to equilibrium, the $A^{(n)}$ are just the orthogonalized product variables,

$$\begin{aligned} A^{(n)}(\mathbf{r}_1 \cdots \mathbf{r}_n) = A^{(1)}(\mathbf{r}_1) \cdots A^{(1)}(\mathbf{r}_n) - \sum_{l=1}^{n-1} \int d\mathbf{r}'_1 \cdots d\mathbf{r}'_l \, d\mathbf{r}''_1 \cdots d\mathbf{r}''_l \\ \times \left\langle A^{(n)}(\mathbf{r}_1 \cdots \mathbf{r}_n) A^{(l)}(\mathbf{r}'_1 \cdots \mathbf{r}'_l)\right\rangle K^l(\mathbf{r}'_1 \cdots \mathbf{r}'_l, \mathbf{r}''_1 \cdots \mathbf{r}''_l) \\ \times A^{(l)}(\mathbf{r}''_1 \cdots \mathbf{r}''_l) \end{aligned} \tag{10}$$

where $K^{(l)}$ is the inverse of the mean square average of $A^{(l)}$,

$$\begin{aligned} \int d\mathbf{r}'_1 \cdots d\mathbf{r}'_l K^{(l)}(\mathbf{r}_1 \cdots \mathbf{r}_l, \mathbf{r}'_1 \cdots \mathbf{r}'_l)\left\langle A^{(l)}(\mathbf{r}'_1 \cdots \mathbf{r}'_l) A^{(l)}(\mathbf{r}''_1 \cdots \mathbf{r}''_l)\right\rangle \\ = \delta(\mathbf{r}'_1 - \mathbf{r}''_1) \cdots \delta(\mathbf{r}'_l - \mathbf{r}''_l) \end{aligned} \tag{11}$$

The derivation[1–5] of Eqs. (1)–(4) in no way depends upon the "usual" choice of A and ϕ. If we regard A and ϕ (unsuperscripted) in Eqs. (1)–(4) as shorthand notation for $A^{(1)} \cdots A^{(m)}$ and $\phi^{(1)} \cdots \phi^{(m)}$, and if we regard $\int d\mathbf{r}$ as implying integration over all necessary spatial variables [as in Eq. (9)], then these equations may be used within GLE for any choice of m. The hydrodynamic matrix, M, is then the multilinear matrix, $M^{(m)}$, describing the coupled dynamics of the set $A^{(1)} \cdots A^{(m)}$. Use of GLE in the NESS formalism should give a better theory than use of LE. The very useful and successful mode-coupling theory[9] is based upon the idea that $A^{(1)}$ does not contain all the slow variables, and products of $A^{(1)}$'s must also be considered slow. Thus use of $A^{(1)} \cdots A^{(m)}$ for $m > 1$ in Eqs.

(1)–(4) gives a more complete collection of slow variables and makes the theory more correct.

To keep the calculation tractable, we choose $m = 2$. We set $\phi^{(1)} = 0$, and, being near equilibrium, we take that part of $\langle x \rangle_{ss}$ linear in $\phi^{(2)}$. Equations (1)–(3) then give

$$\langle x \rangle_{ss} = \int d\mathbf{r}\, d\mathbf{r}' \left[\langle x A^{(2)}(\mathbf{r}, \mathbf{r}') \rangle \phi^{(2)}(\mathbf{r}, \mathbf{r}') - \int_0^\infty dt \langle x(t) \dot{A}^{(2)D}(\mathbf{r}, \mathbf{r}') \rangle \phi^{(2)}(\mathbf{r}, \mathbf{r}') \right] \tag{12}$$

and, schematically, from Eq. (4),

$$\dot{A}^{(2)D} = \dot{A}^{(2)} - M_{22}^{(2)} A^{(2)} - M_{21}^{(2)} A^{(1)} \tag{13}$$

and $M^{(2)}$ is the "bilinear" hydrodynamic matrix of Keyes and Oppenheim.[10]

Thus, with use of GLE, the NESS formalism directly gives a nontrivial answer for $\langle x \rangle_{ss}$ even if $\phi^{(1)} = 0$; this makes perfect sense. As mentioned earlier, in a random ss the fluctuations of the hydrodynamic variables are perturbed and their mean values are left alone. Such a situation cannot be described by LE, but can be described by GLE. It is natural that $\phi^{(2)}$, the potential which determines the second moment, should replace $\phi^{(1)}$ as the key parameter in a random ss.

Let us now compare the results arising from the two approaches. Since we are discussing spatially uniform systems, it is convenient to pass to a Fourier representation of the variables,

$$f_{\mathbf{k}} = \int d\mathbf{r} \exp(i\mathbf{k} \cdot \mathbf{r}) f(\mathbf{r}) \tag{14}$$

for any f. For the same reason, $\phi^{(2)}(\mathbf{r}, \mathbf{r}')$ and $\overline{\phi^{(1)}(\mathbf{r}) \phi^{(1)}(\mathbf{r}')}$ are functions of $\mathbf{r} - \mathbf{r}'$ only, and

$$\begin{aligned} \overline{\phi_{\mathbf{k}}^{(1)} \phi_{\mathbf{k}'}^{(1)}} &= \frac{\delta(\mathbf{k} + \mathbf{k}')}{(2\pi)^3 V} \overline{\phi_{\mathbf{k}}^{(1)} \phi_{-\mathbf{k}}^{(1)}} \\ \phi_{\mathbf{k},\mathbf{k}'}^{(2)} &= \frac{\delta(\mathbf{k} + \mathbf{k}')}{(2\pi)^3 V} \phi_{\mathbf{k},-\mathbf{k}}^{(2)} \end{aligned} \tag{15}$$

where V is the volume.

For the LE terms, we require the relation, which follows from Eq. (10),

$$\begin{aligned} A_{\mathbf{k},\mathbf{k}'}^{(2)} &= A_{\mathbf{k}}^{(1)} A_{\mathbf{k}'}^{(1)} - \langle A_{\mathbf{k}}^{(1)} A_{\mathbf{k}'}^{(1)} A_{-\mathbf{k}-\mathbf{k}'}^{(1)} \rangle \langle A_{\mathbf{k}+\mathbf{k}'}^{(1)} A_{-\mathbf{k}-\mathbf{k}'}^{(1)} \rangle^{-1} A_{\mathbf{k}+\mathbf{k}'}^{(1)} \\ &\equiv A_{\mathbf{k}}^{(1)} A_{\mathbf{k}'}^{(1)} - C_{\mathbf{k},\mathbf{k}'} A'_{\mathbf{k}+\mathbf{k}'} \end{aligned} \tag{16}$$

Thus,

LE term

$$= \begin{cases} \dfrac{1}{2}\dfrac{1}{(2\pi)^3 V}\int d\mathbf{k}\left\langle xA_{\mathbf{k}}^{(1)}A_{-\mathbf{k}}^{(1)}\right\rangle \overline{\phi_{\mathbf{k}}^{(1)}\phi_{-\mathbf{k}}^{(1)}} & [\text{from Eq. (8)}] \\ \dfrac{1}{(2\pi)^3 V}\int d\mathbf{k}\left\langle x\left[A_{\mathbf{k}}^{(1)}A_{-\mathbf{k}}^{(1)} - C_{\mathbf{k},-\mathbf{k}}A_0^{(1)}\right]\right\rangle \phi_{\mathbf{k},-\mathbf{k}}^{(2)} & [\text{from Eq. (12)}] \end{cases} \tag{17}$$

These two expressions will be equivalent, with $\phi_{\mathbf{k},-\mathbf{k}}^{(2)} = (1/2)\overline{\phi_{\mathbf{k}}^{(1)}\phi_{-\mathbf{k}}^{(1)}}$, if the second term in the second line of Eq. (17) vanishes, which is easily shown to be the case. The term is proportional to $\langle xA_0\rangle$, which is proportional to $\partial\langle x\rangle/\partial\phi$, i.e., to an ordinary thermodynamic derivative of $\langle x\rangle$. (These arguments have been discussed in great detail by Oppenheim and co-workers[1-5].) But we can always study x such that $\langle x\rangle = 0$ by simply subtracting $\langle x\rangle$ if nonzero, and the average never contributes anything of interest to our ss calculations. Thus the two methods give the same LE term.

The dissipative terms are more complicated,

dissipative term

$$= \begin{cases} \dfrac{1}{(2\pi)^3 V}\displaystyle\int_0^\infty dt\int d\mathbf{k}\left\langle x(t)\dot{A}_{\mathbf{k}}^{(1)D}A_{-\mathbf{k}}^{(1)}\right\rangle \overline{\phi_{\mathbf{k}}^{(1)}\phi_{-\mathbf{k}}^{(1)}} & [\text{from Eq. (8)}] \\ \dfrac{1}{(2\pi)^3 V}\displaystyle\int_0^\infty dt\int d\mathbf{k}\left\langle x(t)\dot{A}_{\mathbf{k},-\mathbf{k}}^{(2)D}\right\rangle \phi_{\mathbf{k},-\mathbf{k}}^{(2)} & [\text{from Eq. (12)}] \end{cases} \tag{18}$$

So, based on the previous paragraph, a sufficient condition for equality of our two approaches is

$$1/2\left\langle x(t)\dot{A}_{\mathbf{k},-\mathbf{k}}^{(2)D}\right\rangle = \left\langle x(t)\dot{A}_{\mathbf{k}}^{(1)D}A_{-\mathbf{k}}^{(1)}\right\rangle$$

From Eqs. (16) and (13),

$$\begin{aligned} \dot{A}_{\mathbf{k},-\mathbf{k}}^{(2)D} = {} & \dot{A}_{\mathbf{k}}^{(1)}A_{-\mathbf{k}}^{(1)} + A_{\mathbf{k}}^{(1)}\dot{A}_{-\mathbf{k}}^{(1)} - C_{\mathbf{k},-\mathbf{k}}\dot{A}_0 \\ & - M_{22}^{(2)}\left(A_{\mathbf{k}}^{(1)}A_{-\mathbf{k}}^{(1)} - C_{\mathbf{k},-\mathbf{k}}A_0\right) - M_{21}^{(2)}A_0^{(1)} \end{aligned} \tag{19}$$

while

$$\dot{A}_{\mathbf{k}}^{(1)D}A_{-\mathbf{k}}^{(1)} = \dot{A}_{\mathbf{k}}^{(1)}A_{-\mathbf{k}}^{(1)} - M^{(1)}A_{\mathbf{k}}^{(1)}A_{-\mathbf{k}}^{(1)} \tag{20}$$

Arguments similar (for $\dot{A}_0$) and identical (for A_0) to the one used above allow us to ignore A_0 and $\dot{A}_0$ in Eq. (19).

Since our results depend upon an integral over $\mathbf{k}$, and since $\phi^{(2)}_{\mathbf{k},-\mathbf{k}}$ and $\phi^{(1)}_{\mathbf{k}}\phi^{(1)}_{-\mathbf{k}}$ do not depend on the sign of $\mathbf{k}$ (no preferred direction in the system), it is clear that $1/2(\dot{A}^{(1)}_{\mathbf{k}}A^{(1)}_{-\mathbf{k}} + A^{(1)}_{\mathbf{k}}\dot{A}^{(1)}_{-\mathbf{k}})$ gives the same contribution as $\dot{A}^{(1)}_{\mathbf{k}}A^{(1)}_{-\mathbf{k}}$. By the same reasoning, we see that the two theories will be equivalent if

$$M^{(2)}_{\mathbf{k},-\mathbf{k}} = M^{(1)}_{\mathbf{k}} + M^{(1)}_{-\mathbf{k}} \tag{21}$$

We have discussed $M^{(2)}$ in earlier papers[8,9]; evaluation of $M^{(2)}$ is very difficult. However, in the simplest version of bilinear hydrodynamics, where averages involving four $A^{(1)}$'s or $\dot{A}^{(1)}$'s are approximated as all possible products of averages of two $A^{(1)}$'s or $\dot{A}^{(1)}$'s ("factorization approximation"), Eq. (21) is true. In sum, the intuitive adiabatic approach and the more formal approach are equivalent subject to (21), which has already been used in prior work.

3. DISCUSSION OF FORMAL RESULTS

The main goal of the paper has now been accomplished. We have shown how to use the NESS formalism to study random ss. In principle, this provides a rigorous foundation for the study of such systems. Hopefully, the use of two alternate routes, one relatively physical and one more formal, to the final result has helped clarify the physics behind the quantities in GLE, i.e., $\phi^{(2)}$.

The results of Section 2 also clarify the approximations in the usual NESS theories, based upon LE rather than GLE. A GLE theory with $m = \infty$ must be regarded as the most complete version of NESS. Ignoring $\phi^{(n)}$ for $n > 1$ will or will not be valid depending on the details of each problem.

4. WEAKLY STIRRED SYSTEM

As an illustration that our formal theory may be used for real calculations, we now study the "energy cascade" in a randomly stirred fluid; we will ingore all contributions which do not arise via cascade. Since the theory is supposed to hold close to equilibrium, we will obtain results valid for gentle stirring.

A relatively small amount of work has been done in this area. Theories and measurements have been given for[11] the large distance and long time behavior of the velocity correlation function, for[12] the skewness parame-

ter, and for[8] the efficiency of sound generation. However, we have not found a calculation of the cascade effect for gentle stirring. Having shown that the two approaches to random ss given in Section 2 are equivalent, we proceed from Eq. (8), so from here on all quantities are defined in LE, not GLE.

Our goal is to study the "cascade" of energy to high wave vector in a fluid randomly stirred at low wave vector; thus, $\phi_{\mathbf{k}}$ is nonzero for low $\mathbf{k}$ only. It is easy to see that the LE term gives no cascade, i.e., it will produce no energy at wave vectors where $\phi_{\mathbf{k}} = 0$. Thus, we need only consider the dissipative term from here on.

Both for simplicity and to facilitate comparison with the greater part of existing turbulence literature, we consider an incompressible fluid, and, in addition, we ignore temperature fluctuations. So the only conserved variables of interest are the transverse components of the momentum density, $\mathbf{g}$, ϕ is the conjugate velocity field, V, and I^D is the dissipative stress tensor, $\boldsymbol{\sigma}$.

If we choose the wave vector, $\mathbf{k}$ along the $\hat{z}$ axis, then the energy in the motion of the incompressible fluid, $E_{\mathbf{k}}$, is

$$E_{\mathbf{k}} = \frac{1}{2m} \sum_{n=x,y} g^n_{\mathbf{k}} g^n_{-\mathbf{k}} \tag{22}$$

where subscript $\mathbf{k}$ denotes a Fourier component and m is the mass. So, we let $x = E_{\mathbf{k}}$ and use a Fourier representation of $\langle x \rangle_D$ in Eq. (8), with the result

$$\langle E_{\mathbf{k}} \rangle_{ss} = \frac{1}{2m(k_B T)^2} \frac{1}{(2\pi)^3 V} \times \int_0^\infty dt \int d\mathbf{q} \sum_{n=x,y} \left\langle g^n_{\mathbf{k}}(t) g^n_{-\mathbf{k}}(t) g^{\alpha}_{\mathbf{q}} \theta^{\beta\gamma}_{-\mathbf{q}} \right\rangle i q_\gamma \overline{v^\beta_{\mathbf{q}} v^\alpha_{-\mathbf{q}}} \tag{23}$$

with a sum over repeated indices implied.

The form of the turbulent velocity correlation function of an incompressible fluid is well known,[8]

$$\overline{v^\beta_{\mathbf{q}} v^\alpha_{-\mathbf{q}}} = (\delta_{\alpha\beta} - \hat{q}_\beta \hat{q}_\alpha) U(q^2) \tag{24}$$

[$U(q^2)$ is ultimately determined by the stirring force.] Since we wish to confine the stirring to low wave vector,

$$U(q^2) = 0, \qquad q > q_c \tag{25a}$$

Also, for an incompressible fluid[8]

$$U(q^2) \propto q^2, \qquad q \to 0 \tag{25b}$$

Thus, the appropriate form of U for arbitrarily low wave vector stirring is

$$\begin{aligned} U(q^2) &= cq^2, \qquad q < q_c \\ &= 0, \qquad q > q_c \end{aligned} \tag{26}$$

c being a constant. We finally obtain

$$\langle E_{\mathbf{k}} \rangle_{ss} = \frac{ic}{2m(k_B T)^2} \frac{1}{(2\pi)^3 V} \sum_{n=x,y} \int_0^\infty dt \int_0^{q_c} q^3 d\mathbf{q}$$

$$\times \left\langle g_{\mathbf{k}}^n(t) g_{-\mathbf{k}}^n(t) g_{\mathbf{q}}^\alpha \sigma_{-\mathbf{q}}^{\beta\gamma} \right\rangle q_\gamma (\delta_{\beta\alpha} - \hat{q}_\beta \hat{q}_\alpha) \tag{27}$$

Equation (27) is the starting point for our calculation of the energy cascade. We expect Eq. (27) to be valid for weak, slow stirring and $k_c > q_c$ (as we ignore noncascade phenomena). The $\mathbf{k}$ and $\mathbf{q}$ dependent time correlation function describes the cascade; if it is nonzero for $\mathbf{k} \neq \mathbf{q}$, then external driving at wave vector q produces energy at wave vector $\mathbf{k}$.

Note that Eq. (27) is based upon statistical mechanics, while almost all other calculations of the cascade effect start from the Navier–Stokes equation. Of course, there is often no reason to doubt the validity of such approaches. Nevertheless, formulation of a closed form of expression for $\langle E_k \rangle_{ss}$ which is, in principle, more soundly based than the phenomenological theories seems nontrivial.

5. THE TIME CORRELATION FUNCTION

The four-variable time correlation function in Eq. (27) must now be evaluated. This correlation function is somewhat more complicated than those which have entered previous[1–5] (first order in ϕ) NESS calculations, as, by going to second order in ϕ, we have introduced the fourth variable, $g_{\mathbf{q}}^\alpha$. The MIT group has[1–5] given various calculations based upon mode coupling of the simpler correlation functions arising in linear ss.

We now give a systematic method, based upon multilinear hydrodynamics, for the calculation of multitime, multivariable correlation functions. The basic idea for the method is contained in Machta's thesis[5]. We clearly cannot hope to obtain the correlation function exactly, so we focus on the hydrodynamic contribution. This should be appropriate for comparison with more traditional calculations, as any theory of the cascade effect based upon hydrodynamics (as usual) is obviously purely hydrodynamic.

As discussed earlier, it is now well known[5,9,10] that, to exhibit all the long-time, hydrodynamic behavior of a fluid, it is necessary to write equations for the conserved variables and all products thereof, i.e., the multilinear hydrodynamics of the $A^{(n)}$. Mori's formalism[13] may be used

to obtain the multilinear Langevin equation,

$$\frac{\partial}{\partial t} A_{\mathbf{k}}(t) = MA_{\mathbf{k}}(t) + f_{\mathbf{k}}(t) \tag{28}$$

where f is the multilinear random force, and M and A were defined earlier; k is the sum of all the wave vectors for a product variable. The elements of M are labeled[5,11,12] both by the order of the variable and by the value of the intermediate wave vectors. In general, M is a kernel in time, but the fundamental assumption of multilinear hydrodynamics is that M may be treated as a constant for times longer than microscopic. The formal solution of Eq. (28) is

$$A_{\mathbf{k}}(t) = e^{Mt} A_{\mathbf{k}}(0) + \int_0^t d\Upsilon\, e^{M(t-\Upsilon)} f_{\mathbf{k}}(\Upsilon) \tag{29}$$

For any correlation function, we now use Eq. (29) for all the time-dependent linear variables which may appear (a time-dependent nonlinear variable is still a product of linear variables); for the case at hand, these are the two $\mathbf{g}(t)$'s and both time arguments are the same. The linear variables will be coupled to the linear variables (the $\mathbf{g}$'s for an isothermal in compressible fluid) and random forces by $(e^{Mt})_{1,1}$ to the bilinear variables and random forces by $(e^{Mt})_{1,2}$, etc. The 1–2 coupling involves a single intermediate wave-vector sum, the 1–3 coupling a double sum, etc. It is well established,[5,11,12] especially from the careful work[5] in Machta's thesis, that small parameters may be associated with the sums. So, our general strategy will be to keep that part of the correlation function which contains both the effect of interest (cascade effect in this case) and the fewest number of sums.

It is also necessary to know the properties of the new correlation functions generated by use of Eq. (29), where linear variables will either be replaced by a member of A at zero time or by a time-dependent random force. These correlation functions have two types of contributions: "factorization" terms, in which, for some equality of the wave vectors, (Dirac delta) the correlation function is just a product of lower-order correlation functions, and other terms which involve no Dirac delta functions in the wave vector. The factorization terms have been shown[5,9,10] to dominate the wave-vector sums so we retain them alone; this dominance goes hand in hand with the idea of retaining the fewest sums, due to the action of the Dirac deltas. The above set of rules provides a simple practical method for evaluation of multitime, multivariable correlation functions. The method is analogous to recent work[14] by Ronis based upon Kawasaki's graph theory, in the sense that mode coupling and graph theory provide equivalent approaches to problems.

As we shall soon see, the 1–2 coupling gives a cascade effect, so we ignore tri- and higher-linear variables from here on, which, of course, is consistent with our basic equations. It is possible to obtain $(e^{Mt})_{1,1}$ and $(e^{Mt})_{1,2}$ from the "bilinear hydrodynamics" of Keyes and Oppenheim.[10] They considered a compressible fluid, which reduces to the case at hand if the velocity of sound is allowed to approach infinity. However, we proceed directly to the incompressible equations by applying the Mori formalism to a set A consisting of shear modes and products of shear modes. In this case the subtraction in Eq. (10) vanishes, so $A^{(2)} = A^{(1)}A^{(1)}$. Thus, for $k \| z$, the variables are $g_{\mathbf{k}}^x$, $g_{\mathbf{k}}^y$, and $(\hat{e}_i \cdot \mathbf{g}_{\mathbf{k}+\mathbf{k}'} \hat{\mu}_j \cdot g_{-\mathbf{k}'})$, $k' < k_c$, $i, j = 1$ or 2 where the $\hat{e}$ and $\hat{\mu}$ are sets of three orthogonal unit vectors with $\hat{e}_3 \| \mathbf{k} + \mathbf{k}'$ and $\hat{\mu}_3 \| \mathbf{k}'$. The equations are derived just as in Ref. 10, to which we refer the reader for details. We perform a temporal Laplace transform and eliminate the transformed bilinear variables from the equation for the linear variables. The result is

$$
\begin{aligned}
g_{\mathbf{k}}^n(z) = {} & (z + k^2\eta)^{-1}\left[g_{\mathbf{k}}^n(t=0) + f_{\mathbf{k}}^n(z)\right] \\
& + ik\left(\frac{VK_BT}{Nm}\right)(z + k^2\eta)^{-1} \frac{1}{(2\pi)^3} \int_0^{k_c} d\mathbf{k}'(z + k'^2\eta + |\mathbf{k} + \mathbf{k}'|^2\eta)^{-1} \\
& \times \left\{ \left[\hat{k} \cdot \left(\overleftrightarrow{I} - \hat{e}_3\hat{e}_3 \right) \cdot \mathbf{g}_{\mathbf{k}+\mathbf{k}'}(t=0) \right] \right. \\
& \left. \times \left[\hat{n} \cdot \left(\overleftrightarrow{I} - \hat{\mu}_3\hat{\mu}_3 \right) \cdot \mathbf{g}_{-\mathbf{k}'}(t=0) \right] + f_{\mathbf{k},\mathbf{k}'}^n(z) \right\}
\end{aligned}
\tag{30}
$$

where η is the shear viscosity; $f_{\mathbf{k}}^n(z)$ is the Laplace transforms of the random force for $g_{\mathbf{k}}^n$, as is $f_{\mathbf{k},\mathbf{k}'}(z)$ for the bilinear shear mode formed by the product of two square brackets on the right of Eq. (11). Upon inverse Laplace transformation, Eq. (30) gives $g_{\mathbf{k}}^n(t)$ in the form which we have discussed, i.e., Eq. (29).

If Eq. (30) is Laplace inverted and substituted into Eq. (8) for each $\mathbf{g}(t)$, the result may be first classified as a sum of four, linear–linear, linear–bilinear, bilinear–linear, and bilinear–bilinear. The linear–linear term contains no intermediate wave vectors, and so the only factorizations, which we seek, occur for $\mathbf{k} = \mathbf{q}$. However, $\mathbf{k} = \mathbf{q}$ corresponds to no "cascade" so we can ignore the linear–linear term.

The linear–bilinear (L–3) and bilinear–linear (B–L) terms do give a cascade effect, as we now show; thus, we also ignore the bilinear–bilinear term. The L–B and B–L terms can each be further decomposed into four terms, with no random forces, two random forces, and two with one random force. Since the $\mathbf{g}$'s are conserved, the random forces introduce

extra factors of the wave vector and the intermediate wave vector. All of linear and bilinear Navier–Stokes hydrodynamics[5,9,10] is based upon the idea that both wave vectors are small, so we assume that the contributions with no random forces dominate.

The idea just discussed lead to an expression for the correlation function

$$\begin{aligned}
&\left\langle g_{\mathbf{k}}^{n}(t) g_{-\mathbf{k}}^{n}(t) g_{\mathbf{q}}^{\alpha} \sigma_{-\mathbf{q}}^{\beta\gamma} \right\rangle \\
&\quad = ik \frac{Vk_B T}{Nm} e^{-k^2 \eta t} \frac{1}{(2\pi)^3} \int_0^{q_c} d\mathbf{k}' \\
&\qquad \times \Big\{ \mathrm{LT}(z + k^2\eta)^{-1} (z + k'^2\eta + |\mathbf{k} + \mathbf{k}'|^2\eta)^{-1} \\
&\qquad\quad \times \left\langle \hat{k} \cdot \left[\overleftrightarrow{I} - \hat{e}_3(\mathbf{k}) \hat{e}_3(\mathbf{k}) \right] \cdot \mathbf{g}_{\mathbf{k}+\mathbf{k}'} \hat{n} \cdot \left(\overleftrightarrow{I} - \hat{\mu}_3 \hat{\mu}_3 \right) \cdot \mathbf{g}_{-\mathbf{k}'} g_{-\mathbf{k}}^{n} g_{\mathbf{q}}^{\alpha} \sigma_{-\mathbf{q}}^{\beta\gamma} \right\rangle \\
&\qquad\quad - \mathrm{LT}(z + k^2\eta)^{-1} (z + k'^2\eta + |-\mathbf{k} + \mathbf{k}'|^2\eta)^{-1} \\
&\qquad\quad \times \left\langle g_{\mathbf{k}}^{n} \hat{k} \cdot \left[\overleftrightarrow{I} - \hat{e}_3(-\mathbf{k}) \hat{e}_3(-\mathbf{k}) \right] \cdot \mathbf{g}_{-\mathbf{k}+\mathbf{k}'} \hat{n} \cdot \left(\overleftrightarrow{I} - \hat{\mu}_3 \hat{\mu}_3 \right) \right. \\
&\qquad\qquad \left. \cdot \mathbf{g}_{-\mathbf{k}'} g_{\mathbf{q}}^{\alpha} \sigma_{-\mathbf{q}}^{\beta\gamma} \right\rangle \Big\}
\end{aligned} \tag{31}$$

where $\hat{e}_3(\pm\mathbf{k}) \parallel \pm(\mathbf{k} + \mathbf{k}')$, and LT denote an inverse Laplace transform. The evaluation of the time-dependent correlation function has thus been reduced to evaluation of time-dependent averages of four $\mathbf{g}$'s and one $\overleftrightarrow{\sigma}$. This average may be factorized by taking two $\mathbf{g}$'s together and the remaining two $\mathbf{g}$'s with $\overleftrightarrow{\sigma}$, producing one wave-vector delta function. We classify the resulting terms as follows. The first average on the right-hand side of Eq. (31) (containing $g_{-\mathbf{k}}^{n}$) is denoted I, and the second average (with $g_{\mathbf{k}}^{n}$) is denoted II. The $\mathbf{g}$'s are numbered 1–4 from left to right. The groupings and wave-vector equalities are

$$\begin{aligned}
&\text{Ia: } (1,3,\sigma)\,(2,4) && \mathbf{k}' = \mathbf{q} \\
&\text{Ib: } (2,3,\sigma)\,(1,4) && \mathbf{k}' = -\mathbf{k} - \mathbf{q} \\
&\text{IIa: } (1,2,\sigma)\,(3,4) && \mathbf{k}' = \mathbf{q} \\
&\text{IIb: } (1,3,\sigma)\,(2,4) && \mathbf{k}' = \mathbf{k} - \mathbf{q}
\end{aligned}$$

The manipulations to be carried out from here on are straightforward but tedious. We will therefore give the details of evaluation of the contribution of Ia to $\langle E_{\mathbf{k}} \rangle_{\mathrm{ss}}$, and we will then just quote results for the other three factorizations.

6. THE CONTRIBUTION OF THE FACTORIZATIONS TO $\langle E_{\mathbf{k}} \rangle_{ss}$

The factorization denoted Ia is given by the relation

$$\text{Ia} = \left\langle \hat{n} \cdot \left(\overleftrightarrow{I} - \hat{\mu}_3 \hat{\mu}_3 \right) \cdot \mathbf{g}_{-\mathbf{q}} g_{\mathbf{q}}^{\alpha} \right\rangle \times \left\langle \hat{k} \cdot \left[\overleftrightarrow{I} - \hat{e}_3(\mathbf{k}) \hat{e}_3(\mathbf{k}) \right] \cdot \mathbf{g}_{\mathbf{k}+\mathbf{q}} g_{-\mathbf{k}}^{n} \sigma_{-\mathbf{q}}^{\beta\gamma} \right\rangle \frac{(2\pi)^3}{V} \delta(\mathbf{k} - \mathbf{q}) \tag{32}$$

with

$$\hat{\mu}_3 = \hat{q} \tag{33}$$

$$\hat{e}_3(\mathbf{k}) = \frac{\mathbf{k} + \mathbf{q}}{(\mathbf{k} + \mathbf{q})} \tag{34}$$

$\mathbf{k} \| \hat{z}$, and $n = x$ or y. Noting that

$$\left\langle g_{\mathbf{q}}^{\epsilon} g_{-\mathbf{q}}^{\theta} \right\rangle = N m k_B T \delta_{\alpha\epsilon} \tag{35}$$

where $\delta_{\alpha\epsilon}$ is a Kronecker delta, the first averge in Eq. (32) is easily evaluated,

$$\left\langle \hat{n} \cdot \left(\overleftrightarrow{I} - \hat{q}\hat{q} \right) \cdot \mathbf{g}_{-\mathbf{q}} g_{\mathbf{q}}^{\alpha} \right\rangle = N m k_B T \left[\delta_{\alpha n} - (\hat{n} \cdot \hat{q})(\hat{q} \cdot \hat{\alpha}) \right] \tag{36}$$

Machta and Oppenheim have shown[4,5]

$$\lim_{k \to 0} \left\langle g_{\mathbf{k}}^{\epsilon} g_{-\mathbf{k}}^{n} \sigma_0^{\beta\gamma} \right\rangle = N (m k_B T)^2 \times \left(\delta_{\epsilon\beta} \delta_{n\alpha} + \delta_{\epsilon\gamma} \delta_{n\beta} \right) \tag{37}$$

Since k and q are small in our calculation, we assume that Eq. (37) holds for the second average in Eq. (32), in which case

$$\left\langle \hat{k} \cdot \left[\overleftrightarrow{I} - \hat{e}_3(\mathbf{k}) \hat{e}_3(\mathbf{k}) \right] \cdot \mathbf{g}_{\mathbf{k}+\mathbf{q}} g_{-\mathbf{k}}^{n} \sigma_{-\mathbf{q}}^{\beta\gamma} \right\rangle = N (m k_B T)^2 \left\{ \delta_{\beta z} \delta_{\gamma n} + \delta_{\gamma z} \delta_{\beta n} - (\hat{z} \cdot \hat{e}_3) \left[(\hat{e}_3 \cdot \hat{\beta}) \delta_{\gamma n} + (\hat{e}_3 \cdot \hat{\gamma}) \delta_{\beta n} \right] \right\} \tag{38}$$

where we have noted $\hat{k} \| \hat{z}$. Combining Eqs. (27), (31), and (38) we now obtain

$$\langle E_{\mathbf{k}} \rangle_{ss}^{\text{Ia}} = \frac{-N k_B T c}{2(2\pi)^3} k \int_0^{\infty} dt \int_0^{q_c} q^3 \, d\mathbf{q} \, e^{-k^2 \eta t} \text{LT}(z + k^2 \eta)^{-1} \times (z + q^2 \eta + |\mathbf{k} + \mathbf{q}|^2 \eta)^{-1} \sum_{n=x,y} \sum_{\alpha,\beta,\gamma = x,y,z} \left[\delta_{\alpha n} - (\hat{n} \cdot \hat{q})(\hat{q} \cdot \hat{\alpha}) \right] \times \left\{ \delta_{\beta z} \delta_{\gamma n} + \delta_{\gamma z} \delta_{\beta n} - (\hat{z} \cdot \hat{e}_3) \left[(\hat{e}_3 \cdot \hat{\beta}) \delta_{\gamma n} + (\hat{e}_3 \cdot \hat{\gamma}) \delta_{\beta n} \right] \right\} \times q_{\gamma} (\delta_{\alpha\beta} - \hat{q}_{\alpha} \hat{q}_{\beta}) \tag{39}$$

The next step is to evaluate the $\sum_{\alpha,\beta,\gamma}$ in Eq. (39); let that sum be denoted $S_{\mathrm{Ia}}^{(n)}$. The result is

$$S_{\mathrm{Ia}}^{n} = \hat{q}_z\left(1-2\hat{q}_n^2\right) - (\hat{e}_3\cdot\hat{z})\left[(\hat{e}_3\cdot\hat{n})\hat{q}_n + (\hat{e}_3\cdot\hat{q})\left(1-2\hat{q}_n^2\right)\right] \tag{40}$$

which is relatively simple due to substantial cancellation having occurred. Using $\hat{e}_3 = (\mathbf{k}+\mathbf{q})/|\mathbf{k}+\mathbf{q}|$, Eq. (40) becomes

$$S_{\mathrm{Ia}}^{(n)} = \hat{q}_z\left(1-2\hat{q}_n^2\right) - \frac{1+r\hat{q}_z}{1+2r\hat{q}_z+r^2}\left[r\hat{q}_n^2 + (r+\hat{q}_z)\left(1-2\hat{q}_n^2\right)\right] \tag{41}$$

where

$$r = q/k \tag{42}$$

and r is small for the regime described by the theory. We now combine the two contributions to $S_{\mathrm{Ia}}^{(n)}$ over a common denominator and perform $\sum_{n=x,y} S_{\mathrm{Ia}}^{(n)}$; the latter step is most easily done by writing

$$\sum_{n=x,y} S_{\mathrm{Ia}}^{(n)} = \sum_{n=x,y,z} S_{\mathrm{Ia}}^{(n)} - S_{\mathrm{Ia}}^{(z)}$$

with the result

$$\sum_{n=x,y} S_{\mathrm{Ia}}^{(n)} = \frac{r}{1+2r\hat{q}_z+r^2}\left[1+\hat{q}_z^2-2\hat{q}_z^4 + r\hat{q}_z\left(1-\hat{q}_z^2\right)\right] \tag{43}$$

After almost identical manipulations we find

$$\langle E_{\mathbf{k}}\rangle_{\mathrm{ss}}^{\mathrm{IIa}} = \frac{-Nk_BTc}{(2\pi)^3 2}\, k\int_0^\infty dt \int_0^{q_c} q^3 d\mathbf{q}\, e^{-k^2\eta t}\,\mathrm{LT}(z+k^2\eta)^{-1} \times (z+q^2\eta+|-\mathbf{k}+\mathbf{q}|^2\eta)^{-1} \sum_{n=x,y} S_{\mathrm{IIa}}^{(n)} \tag{44}$$

and

$$\sum_{n=x,s} S_{\mathrm{IIa}}^{(n)} = \frac{r}{1-2r\hat{q}_z+r^2}\left[1+\hat{q}_z^2-2\hat{q}_z^4 - r\hat{q}_z\left(1-\hat{q}_z^2\right)\right] \tag{45}$$

The difference between $\langle E_{\mathbf{k}}\rangle_{\mathrm{ss}}^{\mathrm{Ia}}$ and $\langle E_{\mathbf{k}}\rangle_{\mathrm{ss}}^{\mathrm{IIa}}$ may be summarized by saying that the sign of terms odd in $\hat{q}_z$ has been changed. However, a change of variable from $\hat{\mathbf{q}}$ to $-\hat{\mathbf{q}}$ in the angular part of $\int d\mathbf{q}$ then shows that both contributions are identical, and

$$\langle E_{\mathbf{k}}\rangle_{\mathrm{ss}}^{\mathrm{Ia}} + \langle E_{\mathbf{k}}\rangle_{\mathrm{ss}}^{\mathrm{IIa}} = Nk_BTck\frac{1}{(2\pi)^3}\int_0^\infty dt\int_0^{q_c} q^3 d\mathbf{q}\, e^{-k^2\eta t} \times \mathrm{LT}(z+k^2\eta)^{-1}(z+q^2\eta+|\mathbf{k}+\mathbf{q}|^2\eta)^{-1}\frac{r}{1+2r\hat{q}_z+r^2} \times\left[1+\hat{q}_z^2-2\hat{q}_z^4+r\hat{q}_z\left(1-\hat{q}_z^2\right)\right] \tag{46}$$

Proceeding similarly, we evaluate Ib and IIb, combine them with the change of variable $\hat{q} \to -\hat{q}$, and obtain

$$\langle E_{\mathbf{k}} \rangle_{ss} = 2Nk_B Tck \frac{1}{2(\pi)^3} \int_0^\infty dt \int_0^{q_c} q^5 \, d\mathbf{q} \int d\hat{q} \, e^{-k^2\eta t} \times \frac{\mathrm{LT}(z + k^2\eta)^{-1}(z + q^2\eta + |\mathbf{k} + \mathbf{q}|^2\eta)^{-1}}{1 + 2r\hat{q}_z + r^2} \times \left[(1 + r^2)\hat{q}_z(1 - \hat{q}_z^2) + r(1 + \hat{q}_z^2 - 2\hat{q}_z^4)\right] \tag{47}$$

The Lt is easily done by noting that the LT of a product of functions of z is just a contribution in time of the LT's of the functions. We take the LT and do the time integral, with the result

$$\langle E_{\mathbf{k}} \rangle_{ss} = \frac{2Nk_B TcV}{\eta^2 k^3 (2\pi)^3} \int_0^{q_c} q^5 \, dq \int d\hat{q} \frac{\left[(1 + r^2)\hat{q}_z(1 - \hat{q}_z^2) + r(1 + \hat{q}_z^2 - 2\hat{q}_z^4)\right]}{(1 + 2r\hat{q}_z + r^2)(2 + r^2)(3 + 2r\hat{q}_z + 2r^2)} \tag{48}$$

Finally, we introduce dimensionless variables,

$$\langle E_{\mathbf{k}}^* \rangle_{ss} = \langle E_{\mathbf{k}} \rangle_{ss} / (\tfrac{3}{2} Nk_B T) \tag{49}$$

so E^* is the ratio of the average in ss to the thermal average,

$$x = k/q_c \tag{50}$$

and

$$y = q/q_c \tag{51}$$

$$r = y/x \tag{52}$$

we perform the integral over the azimuthal angle, and find

$$\langle E_x^* \rangle_{ss} = \frac{R^2}{18\pi^2 x^3} \int_0^1 dy \int_{-1}^1 du \frac{(1 + r^2)u(1 - u^2) + r(1 + u^2 - 2u^4)}{(1 + 2ru + r^2)(1 + r^2/2)(1 + \frac{2}{3}ru + \frac{2}{3}r^2)} \tag{53}$$

where R is the Reynolds number, defined as follows: The function, $U(q^2)$, is, besides angular factors, the mean square velocity/wave vector volume element, and, furthermore, is $\propto q^2$. So, we may write

$$U(q^2) = v^2/q_c^3 (q/q_c)^2 \tag{54}$$

where v is a measure of the typical velocity amplitude. The coefficient, c, is then determined,

$$c = v^2/q_c^5 \tag{55}$$

and Eq. (53) then follows from the definition,

$$R = v/\eta(q_c^{-1}) \tag{56}$$

which is the usual definition[8] of R, if one recognizes that q_c^{-1} is the characteristic length for the problem.

7. DISCUSSION OF EXAMPLE

Equation (53) is our final result; it provides a prediction for the energy spectrum in a gently stirred system. The R^2 dependence is expected.[8] The very large x and small x behavior of $\langle E_x \rangle_{ss}$ follows from Eq. (53) by inspection,

$$\langle E_x \rangle_{ss} \propto R^2 \begin{cases} x^2, & x \to 0 \\ x^{-4}, & x \to \infty \end{cases}$$

although, of course, the theory is not to be trusted for small x (large r). Nevertheless, the x^2 behavior is comforting, being required for an incompressible fluid. The large-x decay is x^{-4}, not x^{-3}, because the angular integral in Eq. (53) vanishes when $r = 0$ and is $O(r)$ for small r, or large x. A numerical computation of $\langle E_x \rangle_{ss}$ for the full range of x is given in Fig. 1.

Obviously, the treatment of a weakly stirred system given here appears elaborate when compared to a traditional approach based upon the Navier–Stokes equation. We emphasize that the sample calculation is not

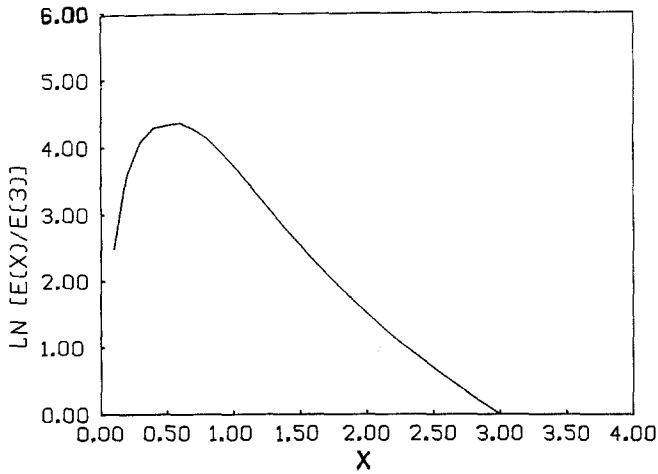

Fig. 1. Plot of $\ln[E(x)/E(3)]$ vs. x, where E is the energy density and x the dimensionless wave vector.

the crux of this paper; that status is reserved for the general NESS theory of random ss. Nevertheless, once "hydrodynamic" approximations are made wherever possible, the hydrodynamic version of the NESS theory is in fact quite simple. In sum, the NESS theory should allow reproduction and justification of traditional results where appropriate, while holding out the potentiality of treating systems where a phenomenological approach is inadequate.

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